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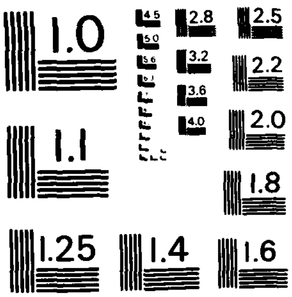

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The study of stochastic wave propagation in the ocean by the method of progressing waves is introduced. The technique is applied to wave propagation through time-dependent random media.

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METHOD OF PROGRESSING WAVES IN TIME-DEPENDENT RANDOM MEDIA

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1. INTRODUCTION

In the wave propagation in randomly fluctuating ocean, most researchers are concerned with the time-independent random media [1], i.e. the local sound speed varies independently of time. For short range propagation, this is perhaps not a serious omission. However, in the long-range transmission, the temporal fluctuation of media becomes important and should be taken into account.

The main objective of our work is to investigate the statistical laws in the acoustic wave propagation through random media, such as the ocean. We introduce the method of progressing waves for the time-dependent random media. If the fluctuation is weak, this is the case in ocean, we are interested in the accumulative effect over a long time. This consideration leads to the diffusions limit: $\varepsilon \to 0$, $t \to \infty$ with $\tau = \varepsilon^2 t$ fixed, where ε is the scale of random fluctuation. Our major result is that, in the diffusion limit, the statistics of the phase and the amplitude for each wave component may be computed easily according to the probability law of a simple diffusion process with the variance parameter σ depending on the covariance function, the invariance principle of Donsker in the classical situation.

In Section 2, we develop the basic theory via a simple model equation. The first-order random PDE (partial differential equation) is to be solved by the method of characteristics. Then we apply the Khasiminski's [2,3] limit theorem to random ordinary DE for the characteristics. Thereby, in the diffusion limit, we can determine the fluctuations of phase and amplitude by the limiting diffusion process. In the next section, the system governs the acoustic waves in one dimension is treated. There we introduce the method of progressing wave solution. This together with the method of characteristics

enable us to resolve the question of statistical fluctuations of waves. Finally, in Section 4, the results in one dimension will be extended to the physically important, three-dimensional problem.

2. A SIMPLE MODEL EQUATION

To illustrate the basic ideas, we consider the following simple, first-order partial differential equation with random coefficient

$$[\frac{\partial}{\partial t} + V(t,x)\frac{\partial}{\partial x}]\ U = \eta_\varepsilon(t,x,\omega)U \tag{2.1}$$

$$U|_{t=0} = a(x), \tag{2.2}$$

where a, V are the given functions and η_ε is a random function depending on a small parameter $\varepsilon > 0$.

By the method of characteristics in partial differential equations, we introduce the characteristic equation

$$\frac{dy}{ds} = V(x,y)\ , \quad Y(t) = x\ , \tag{2.3}$$

whose solution is a characteristic curve Γ_t through (t,x)

$$y = \xi(s;t,x)\ , \quad 0 \leq s \leq t\ . \tag{2.4}$$

Along Γ_t the system (2.1) and (2.2) becomes

$$\frac{d}{ds}\ U[s,\xi(s;t,x)] = \eta_\varepsilon[s,\xi(s;t,x,\omega)]U[s,\xi(s;t,x)]\ . \tag{2.5}$$

$$U[0,\xi(0;t,x)] = a[\xi(0;t,x)]\ . \tag{2.6}$$

The solution of the above simple ordinary differential equation is

$$U(s;t,x,\omega) = a[\xi(t,x)]\exp\ \{\int_s^t \eta_\varepsilon[\sigma;\xi(\sigma;t,x)]d\sigma\}\ , \tag{2.7}$$

where $\xi(t,x) = \xi(0;t,x)$.

From (2.7) the solution of (2.1)-(2.2) is given by

(2.8) $$U(t,x,\omega) = U(0;t,x,\omega) = a[\xi(t,x)]\exp\{\theta_\varepsilon(t,x,\omega)\} ,$$

where the amplitude $a[\xi(t,x)]$ is deterministic, while the phase function $\theta_\varepsilon(t,x,\omega) = \theta_\varepsilon(s;t,x,\omega)|_{s=0}$ with

(2.9) $$\theta_\varepsilon(s,t,x,\omega) = \int_s^t \eta_\varepsilon[\sigma,\xi(\sigma;t,x),\omega]d\sigma ,$$

is random.

For fixed (t,x), the dependence on which will be suppressed, (2.9) takes the form

(2.10) $$\theta_t^\varepsilon(\omega) = \int_s^t \tilde{\eta}_\varepsilon(s,\omega)ds .$$

In studying the statistical law of the solution (2.8), one naturally raises the question, as $\varepsilon \downarrow 0$, $(t-s) \uparrow \infty$, whether a family of processes $\{\theta_t^\varepsilon, t \geq 0\}$ converges weakly to a certain limit distribution $\theta_t(\omega)$, under appropriate scaling. The weak convergence of θ^ε to θ will be denoted by

(2.11) $$\theta^\varepsilon \Rightarrow \theta .$$

The expression (2.10) is in the form of the "sum" of random variables. If the necessary moments exists and the process $\tilde{\eta}_\varepsilon(t,\omega)$ is asymptotically independent, then the normalized $\{\theta_t^\varepsilon - E\,\theta_t^\varepsilon\}/\sigma\{\theta_t^\varepsilon\} = \hat{\theta}_t^\varepsilon \Rightarrow \hat{\theta}_t \in N(0,1)$, the normal distribution with mean zero, variance one. It follows the phase fluctuation is asymptotically normal with mean $E\theta_t$ and variance $\sigma^2\{\theta_t\}$. Thus the wave function satisfies the log-normal law.

Another interesting case is the diffusion limit. This limit is important because it renders the problem computable and permits generalization to higher-dimensional situations. Suppose $\tilde{\eta}_\varepsilon$ is such that

(2.12) $$\tilde{\eta}_\varepsilon = \tilde{\eta}_0 + \varepsilon\tilde{\eta}$$

where $\tilde{\eta}_0$ is deterministic, $E\tilde{\eta} = 0$ and

(2.13) $$E\{\tilde{\eta}(t)\tilde{\eta}(s)\} = a(t,s).$$

Assume the limit

(2.14) $$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} a(t,s)dtds$$

exists, independent of t_0, and the process is asymptotically independent (to be made precise later). Then the rescaled phase function $\Theta^\varepsilon(\tau) = \theta^\varepsilon(\tau/\varepsilon^2)$, for fixed $\tau = \varepsilon^2 t$, we have

$$\Theta^\varepsilon(\tau) \Rightarrow \Theta^\circ(\tau) \quad ,$$

which is a Wiener process with the variance parameter σ. This is a consequence of the well-known Khasiminski theorem [3].

Now we consider a variant of (2.1):

(2.15) $$[\frac{\partial}{\partial t} + \eta^\varepsilon(t,x,\omega) \frac{\partial}{\partial x}]U = b(t,x)\, U ,$$

(2.16) $$U|_{t=0} = a(x) \; .$$

In contrast with the previous equation (2.1), the characteristic equation

(2.17) $$\frac{dy^\varepsilon}{ds} = \eta^\varepsilon(s,y^\varepsilon,\omega) \;\; , \;\; y^\varepsilon(t) = x \; ,$$

is stochastic. The limit theorem for the stochastic differential equation (2.17) was treated by Khasiminski as mentioned before.

Let

(2.18) $$\eta^\varepsilon(t,x,\omega) = \eta_0 + \varepsilon\eta(t,x,\omega)$$

where η_0 is a constant (in general, may be a function x,t). The random field satisfies

(i)

(2.19) $$E\eta = 0$$

(ii) $\eta(t,x,\omega)$ is (a.e.) continuous, and its partial derivatives up to second-order are uniformly bounded in $[0,T] \times \mathbb{R}$.

(iii)

(2.20) $$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \iint_0^T E\eta(t_0+T-t,\; x_0-\eta_0 t)\eta(t_0+T-s, x_0-\eta_0 s)dt$$

which exists independently of $t_0 \geq 0$ and x_0.

(iv) The σ-field F_s^t is increasing in [t,s] such that $\eta(t,x,\omega)$ is measurable with respect to F_t^t, and for all A in $B \in F_{t+T}^{\infty}$, we have

$$|P(A\ B) - P(A)P(B)| \leq \beta(T)P(A)\ . \tag{2.21}$$

where $T^6\beta(T) \downarrow 0$ as $T \uparrow \infty$.

Then we can apply Theorem K (Khasiminski's theorem) to get

Lemma 2.1: As $\varepsilon \downarrow 0$, $(t-s) \uparrow \infty$, with $\tau = \varepsilon^2 t$ and $\sigma = \varepsilon^2 s$ fixed, the solution (2.17) converges weakly to a Markov diffusion about the mean characteristic, that is

$$Y_s^{\varepsilon}(t) \Rightarrow \xi_0(x,t-s) + \xi(\tau-\sigma) = Y_s(t) \tag{2.22}$$

where $\xi_0(x,t) = (x-\eta_0 t)$, $\xi(0,\omega) = 0$,

and $\xi(\tau,\omega)$ is a Markov diffusion process with mean zero and the diffusion coefficient σ^2 defined by (2.20)#.

Remark: In fact the above lemma says that the limit process will satisfy the Itô equation

$$dy_s = \eta_0\, ds + \varepsilon\sigma dw(t-s)\ ,\quad y_t = x. \tag{2.23}$$

Let $\Gamma_t(\omega)$ be the random characteristic curve

$$y^{\varepsilon} = Y_s^{\varepsilon}(t,\omega),\quad 0 \leq s \leq t\ . \tag{2.24}$$

Similar to (2.8), along $\Gamma_t(\omega)$, the system (2.15)-(2.16) may be integrated to give

$$U^{\varepsilon}(t,x) = a[Y_0^{\varepsilon}(t)]\exp\{\phi(t,y^{\varepsilon})\}, \tag{2.25}$$

where

$$\phi(t,y^{\varepsilon}) = \int_0^t b[s,Y_s^{\varepsilon}(t)]ds\ . \tag{2.26}$$

To ensure the weak convergence of the amplitude and the phase in (2.25), we assume the following conditions hold:

$$(2.27)\qquad \lim_{h\downarrow 0}\ \sup_{\varepsilon\in[0,\varepsilon_0]} P\{\sup_{|s'-s|\le h} |Y_0^\varepsilon(s') - Y_0^\varepsilon(s)| > \delta\} = 0,\ \forall\delta > 0$$

$$(2.28)\qquad \lim_{h\downarrow 0}\ \sup_{\varepsilon\in[0,\varepsilon_0]} P\{\sup_{\substack{0\le s,s'\le t\\ |s'-s|<h}} |Y_{s'}^\varepsilon(t) - Y_0^\varepsilon(s)| > \delta\} = 0,\ \forall\delta > 0$$

Then by invoking Theorem 1 on p. 449 and Theorem 2 on p.486 in [4], we can prove the following lemmas.

Lemma 2.2 As $\varepsilon \downarrow 0$ and $t \uparrow \infty$, with $\tau = \varepsilon^2 t$ held fixed, we have $a[Y_0^\varepsilon(t)] \Rightarrow a[\zeta_0(x,t) + \xi(\tau)]$ asymptotically provided that the condition (2.27)) holds and $a \in C(\mathbb{R})$.

Lemma 2.3 Suppose the condition (2.28) holds and if $\exists$ g(x) on $\mathbb{R}$ which is positive, increasing as $|x| \to \infty$ such that

$$\sup_{\varepsilon\in[0,\varepsilon_0]}\ \sup_{t\ge s\ge 0} E\, g[Y_s^\varepsilon(t)] = C < \infty$$

If b satisfies

$$\lim_{N\to\infty}\ \sup_{t\ge 0}\ \sup_{|x|>N} \frac{|b(t,x)|}{g(x)} = 0$$

then

$$\phi(t,Y^\varepsilon) \Rightarrow \phi(t,Y) \text{ asymptotically } \# .$$

Thus the above results may be summarized as

Theorem 2.1. If the assumptions in Lemmas 2.1 and 2.2 are satisfied, then the statistics of the amplitude a and the phase ϕ in the solution (2.25), in the limit as $\varepsilon \downarrow 0$, $t \uparrow \infty$ with $\tau = \varepsilon^2 t$ fixed, may be computed according to the distribution $Y_s(t)$ given in (2.22) #.

Remarks:

(1) In general it is not necessarily true that $U^{\varepsilon} \Rightarrow U = a(Y_0)\exp\{\phi(t,y)\}$. This is valid if we can show that a stronger convergence, such as the convergence in probability, takes place.

(2) Theorem 2.1 implies an invariance principle. For small ε and large t, one can approximately compute the statistics of a and ϕ by means of a Brownian motion (2.22).

(3) "$a[Y_0^{\varepsilon}(t)] \Rightarrow a[Y_0(t)]$ asymptotically" means that, writing $Y_0^{\varepsilon}(t) = \xi_0(x,t) + \xi^{\varepsilon}(\tau)$, the limit is taken with respect to ε, with ξ_0 and τ both held fixed. Similar interpretation is given for $\phi(t,Y^{\varepsilon})$.

3. RANDOM HYPERBOLIC SYSTEMS IN ONE SPACE-DIMENSION

The acoustic wave propagation in one-dimension may be described by the first-order system:

$$\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial u}{\partial x} = q_1 , \tag{3.1}$$

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = q_2 ,$$

which is subject to appropriate initial and boundary conditions. In (3.1), p and u denote the acoustic pressure and velocity; ρ is the mean density; $c(t,x,\omega)$ is the local speed of sound which fluctuates randomly, and q_1, q_2 are the source terms. For convenience, we let $U_1 = \rho u$, $U_2 = p$ so that (3.1) can be written as

$$\frac{\partial U}{\partial t} - A\frac{\partial U}{\partial x} = Q , \tag{3.2}$$

where

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} ,$$

$$A = -\begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix} , \tag{3.3}$$

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} .$$

In the theory of partial differential equations [5], the system (3.2) is usually analyzed by the method of characteristics. But this approach yields a rather complicated statistical problem. Such a consideration has lead us to a formal progressing wave solution. Though difficult to justify mathematically, it is physically appealing and has been applied widely in studying time-dependent wave propagation problems. More importantly the method may be easily extended to higher dimensions.

Let us consider the homogeneous version of (3.2):

$$L\,U = \frac{\partial U}{\partial t} - A\,\frac{\partial U}{\partial x} = 0\ . \tag{3.4}$$

For the first-order progressing wave solution, we seek a solution of the form (p.622, [5])

$$U(t,x) \sim \sum_{j=1}^{N} z_j(t,x)\,g_j[S(t,x)] \tag{3.5}$$

where z is the amplitude-vector; g is the scalar wave form, and S is the phase function.

Let

$$U_j = z_j g_j(S)\ , \quad j = 1,2,\ldots\ . \tag{3.6}$$

Then

$$\begin{aligned} L\,U_1 &= g_1\,Lz_1 + g_1[S_t I - S_x A]\,z_1 \\ &= g_1\,LZ, \end{aligned} \tag{3.7}$$

if we choose

$$Mz_1 = (S_t I - S_x A)\,z_1 = 0 \tag{3.8}$$

Let λ and z^+ denote the eigenvalue and the right eigenvector of $(S_x A)$, respectively. We get

$$z_j = \rho_j z_j^+\ , \quad j = 1,2,\ldots \tag{3.9}$$

(3.10) $$S_t = \lambda(x,t,S_x) .$$

Thus

(3.11) $$M = (\lambda I - S_x A) ,$$

and

(3.12) $$L\, U_2 = g_2' M z_2 + g_1 L z_2 .$$

It is easy to check that

(3.13) $$L(U_1 + U_2) = g_1 L z_2$$

if we choose

(3.14) $$g_2' = g_1$$

and

(3.15) $$M z_2 = -L z_1 \quad .$$

This process can be continued in such a manner that

(3.16) $$L\left(\sum_{j=1}^{n} U_j\right) = g_1 L\, z_n, \quad \text{for any } n \leq N.$$

We tacitly assumed that the sequence $\{z_n\}$ is decreasing in magnitude so that the remainder term on the RHS of (3.16) also diminishes. For simplicity, we stop at $n = 2$. In view of (3.9), (3.15) becomes

(3.17) $$M z_2 = -L(\rho_1 z^+)$$

which is solvable provided that

(3.18) $$z^- L(\rho_1 z^+) = 0 .$$

Here z^- denotes a left eigenvector of $(S_x A)$. Hence, to a first-order approximation, the amptitude and the phase of the wave function U are determined by (3.10) and (3.18).

According to the Hamilton-Jacobi theory, we introduce the "bi-characteristic" curve $\tilde{\Gamma}_t$:

(3.19) $$\frac{dy}{ds} = -\frac{\partial}{\partial q}\lambda(s,y,q), \quad y(t) = x,$$

(3.20) $$\frac{dq}{ds} = \frac{\partial}{\partial y}\lambda(s,y,q), \quad q(t) = p = S_x .$$

The curve $\Gamma_t = Y = Y(x,s)$, $0 \le s \le t$, is the physical characteristics or the ray. Note that if λ is linear in q, then (3.19) is independent of (3.20). The amplitude or "transport" equation (3.18) can be simplified to give

$$(3.21) \qquad \left(\frac{\partial}{\partial t} - \tilde{\lambda}\frac{\partial}{\partial x}\right)\rho + b\rho = 0, \qquad \rho(0,x) = a(x)$$

where $\rho = \rho_1$ and $\tilde{\lambda} = \lambda/S_x$.

$$(3.22) \qquad b = (z^- L z^+) \quad \text{with } (z^-, z^+) = 1$$

Now the system (3.19)-(3.21) resembles the system (2.1)-(2.3) treated in section 2. Therefore the analysis there may be carried over to the present case.

To illustrate the procedure, we return to the original problem (3.2) with Q = 0. It is easy to see that there are two distinct eigenvalues

$$(3.23) \qquad \lambda = \lambda_{1,2} = \pm C(t,x,\omega)S_x(t,x,\omega)$$

The corresponding normalized eigenvector are

$$(3.24) \qquad z^+_{1,2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\lambda_{1,2} \end{bmatrix} \text{ and } z^-_{1,2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \lambda^{-1}_{1,2} \end{bmatrix} .$$

The equation (3.10), (3.19) and (3.21), become

$$(3.25) \qquad S_t = \pm C(t,x,\omega)S_x , \quad S(0,x) = \theta(x) ,$$

$$(3.26) \qquad \frac{dy}{ds} = \pm C(s,y,\omega), \quad y(t) = x.$$

$$(3.27) \qquad \rho_t \pm C\rho_x + b\rho = 0, \quad \rho(0,x) = a(x)$$

where $b = (z^-_{1,2} L z^+_{1,2})$.

Now suppose that the local sound speed C fluctuates about a constant mean value η_0 so that

$$(3.28) \qquad C(t,x,\omega) = \eta_0 + \varepsilon\eta(t,x,\omega)$$

where $\varepsilon > 0$ is a small parameter. Thus for each characteristic curve, we can

apply Theorem 2.1 to (3.26) to obtain an asymptotic limit law along the unperturbed characteristic curve, or, as $\varepsilon \downarrow 0$, $t \uparrow \infty$, $\tau = \varepsilon^2 t$ fixed,

(3.29) $$Y^{\varepsilon}_{1,2}(s;x,t) \Rightarrow Y_{1,2}(s;x,t) \text{ asymptotically,}$$

where $Y(s;x,t)$ is the Brownian motion defined by the Itô equation (2.23).

In view of (3.28), since L is a differential operator, $b = O(\varepsilon)$. If we neglect b in (3.27), the equations (3.25) and (3.27) become identical. Their solutions are simply

(3.30) $$S(t,x,\omega) = \theta_j[Y^{\varepsilon}_j(x,t,\omega)] \ ,$$

(3.31) $$\rho(t,x,\omega) = a_j[Y^{\varepsilon}_j(x,t,\omega)] \ , \ Y^{\varepsilon}_j(x,t) = Y^{\varepsilon}_j(0;x,t), \ j=1,2 \ .$$

Suppose we assume that η, a, θ satisfy the conditions for Lemma 2.2. Then we can assert that

(3.32) $$\theta_j[Y^{\varepsilon}_j(x,t)] \Rightarrow \theta_j[Y_j(x,t)]$$

and

(3.33) $$a_j[Y^{\varepsilon}_j(x,t)] \Rightarrow a_j[Y_j(x,t)] \ , \quad j = 1,2 \ , \text{ asymptotically.}$$

That is, for small ε, we may compute the statistics of amplitude and phase for each wave component by the distribution of the Brownian motion $Y_{1,2}(0,t)$.

To write down pathwise solution explicitly, let the initial state $U(0,x)$ be written as

(3.34) $$U(0,x) = \alpha_1 a_1(x) z_1^{+} \exp\{ik\theta_1(x)\} + \alpha_2 a_2(x) z_2^{+} \exp\{ik\theta_2(x)\} \ .$$

Then the first-order progressing wave approximation is

(3.35) $$U(t,x,\omega) \sim \alpha_1 a_1[Y^{\varepsilon}_1(x,t)] z_1^{+} \exp\{ik\theta_1[Y^{\varepsilon}_1(x,t)]\} + \alpha_2 a_2[Y^{\varepsilon}_2(x,t)] z_2^{+} \exp\{ik\theta_2[Y^{\varepsilon}_2(x,t)]\} \ .$$

In view of (3.24), the eigenvectors $z^{+}_{1,2}$ are random. But from (3.28), we have

$$(3.36) \qquad z^{+}_{1,2} = C_{1,2} + O(\varepsilon), \; C_{1,2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mp\eta_0\theta_x(x\pm\eta_0 t) \end{bmatrix}$$

where C is not random. Thus z_j may be replaced by C_j in (3.35).

Let us summarize the above results as a theorem.

<u>Theorem 3.1</u>: In the first-order progressing wave approximation, the acoustic problem (3.1) or (3.2), with $q=0$ and the initial condition (3.34) has a sample solution (3.35), where z_j may be replaced by C_j, $j=1,2$. Thus, in the diffusion limit $\varepsilon \downarrow 0$, $t \uparrow \infty$, with $\tau = \varepsilon^2 t$ fixed, the statistics of the amplitude a_j and the phase θ_j, $j = 1,2$, can be computed approximately by the probability laws for the Brownian motions $Y_j(x,t)$ satisfying

$$(3.37) \qquad dY_j = (-1)^j \mu_0 dt + \sigma dw(t), \quad Y_j(x,0) = x, \; j = 1,2,$$

where $w(t)$ is the standard Brownian motion and σ is defined by (2.20). #

4. RANDOM HYPERBOLIC SYSTEMS IN HIGHER DIMENSIONS

In three dimensions the acoustic system (2.1) in one dimension should be modified to give

$$\frac{\partial p}{\partial t} + \rho c^2 \, \nabla \cdot u = q_1 \, ,$$

$$\rho \frac{\partial u}{\partial t} + \nabla p = q_2 \, .$$

Here $x = (x_1,x_2,x_3)$, $u = (u_1,u_2,u_3)$ and $q_2 = (q_{21},q_{22},q_{23})$, and they have similar interpretation as before. For convenience, we set

$$(4.2) \qquad \partial_t = \frac{\partial}{\partial t} \, , \; \partial_j = \frac{\partial}{\partial x_j} \, , \quad j = 1,2,3 \, .$$

Define

$$(4.3) \qquad U = \begin{bmatrix} \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ p \end{bmatrix} ,$$

$$(4.4) \qquad A_j = (-1)\begin{bmatrix} 0 & 0 & 0 & \delta_{1j} \\ 0 & 0 & 0 & \delta_{2j} \\ 0 & 0 & 0 & \delta_{3j} \\ C^2\delta_{1j} & C^2\delta_{2j} & C^2\delta_{3j} & 0 \end{bmatrix}, \quad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases} \; i,j=1,2,3.$$

$$(4.5) \qquad Q = \begin{bmatrix} q_1 \\ q_{21} \\ q_{22} \\ q_{23} \end{bmatrix} .$$

Then (4.1) may be put in the form

$$(4.6) \qquad L\,U = (\partial_t - \sum_{j=1}^{3} A_j \partial_j)\,U = Q .$$

which is *a generalization* of (3.2). By an analogy to one-dimensional case, it is clear that the progressing wave formalism can be extended to the present case in a straight-forward manner. Therefore we shall only sketch the procedure without going into the details.

In the first-order approximation, as before, let

$$(4.7) \qquad U \sim zg[S(t,x)] .$$

Then we set $Q = 0$ in (4.6),

$$(4.8) \qquad LU = gLz ,$$

and

$$(4.9) \qquad Mz = [S_t] - \sum_{j=1}^{3} (\partial_j S) A_j] z = 0 .$$

The matrix M has two repreated eigenvalues $\lambda = 0$ and two distinct ones at

$$(4.10) \qquad \lambda_{1,2} = \pm\gamma C ,$$

where

$$\gamma = \{\sum_{j=1}^{3} (\partial_j S)^2\}^{\frac{1}{2}} \tag{4.11}$$

Again denote by $z^+_{1,2}(z^-_{1,2})$ the right (left) eigenvector corresponding to $\lambda_{1,2}$, respectively, with $(z^-,z^+) = 1$.

Suppose that the initial wave vector belongs to the subspace spanned by z^+_1, z^+_2 so that

$$U(0,x) = \sum_{j=1}^{2} \alpha_j a_j(x) z^+_j(0,x) \exp\{ik\theta_j(x)\} . \tag{4.12}$$

Then, corresponding to (3.25)-(3.27), the amplitude and the phase are determined by the following set of equations

$$\partial_t S = \pm C(t,x,\omega)|\nabla S|, \ S(0,x) = \theta(x) \tag{4.13}$$

which has the bi-characteristic equations:

$$\frac{dy}{ds} = \pm C(s,y,\omega)\frac{q}{|q|}, \ q \neq 0, \ y(t) = x , \tag{4.14}$$

$$\frac{dq}{ds} = \pm\nabla C(s,y,\omega)|q| , \qquad q(t) = p = \nabla S . \tag{4.15}$$

and

$$\partial_t \rho \mp \frac{C}{|\nabla S|} \nabla S \cdot \nabla \rho + b\rho = 0, \ \ \rho(0,x) = a(x) . \tag{4.16}$$

In contrast with the one-dimensional case, the system of equations (4.13)-(4.16) are highly nonlinear. Even without randomness, the system is rather difficult to deal with. In order to proceed, we shall linearize the system. Again assume

$$C(t,x,\omega) = \eta_0 + \varepsilon\eta(t,x,\omega). \tag{4.17}$$

Then (4.15) implies $q \cong p$, and (4.14) yields

$$\frac{dy}{ds} = \pm C(s,y,\omega)\hat{p}, \ y(t) = x, \tag{4.18}$$

$$\hat{p} = p/|p| = \nabla S_0/|\nabla S_0| , \tag{4.19}$$

where $S_0(t,x)$ denotes the phase function as $\varepsilon = 0$. The "eikonal" equation (4.13) may be approximated by

(4.20) $$\partial_t S = \pm\, C(t,x,\omega)\hat{p} \cdot \nabla S, \quad S(0,x) = \theta(x) \ .$$

In the same spirit, (4.16) is replaced by

(4.21) $$\partial_t \rho \mp C\, \hat{p} \cdot \nabla\rho = 0, \quad \rho(0,x) = a(x)$$

Now the linearized system (4.18) - (4.21) is similar to (3.25) - (3.27) in one-dimension, and can be treated accordingly. We are able to generalize Theorem 3.1 to the following.

Theorem 4.1: In the first-order progressing wave approximation, the acoustic wave problem (4.1) or (4.6), with $Q = 0$ and the initial condition (4.12), has a sample solution of the form

(4.22) $$U(t,x,\omega) \sim \sum_{j=1}^{2} \alpha_j a_j [y_j^\varepsilon(x,t)] \exp\{ik\ \theta_j [y_j^\varepsilon(x,t)]\}$$

where $a_j = Z_j^+ |_{\varepsilon=0}$.

Further, in the diffusion limit, the statistics of the amplitude a_j and the phase θ_j can be determined by the probability distribution of the Brownian motion $Y_j(x,t)$ which satisfies

(4.23) $$dy_j = [(-1)^j \mu_0 dt + \sigma\, dw(t)]\hat{p}_0 \ ,$$

$$y_j(x,0) = x, \quad j = 1,2,$$

where $\hat{p}_0 = \dfrac{\nabla\theta(x)}{|\nabla\theta(x)|}$, $w(t)$ is the standard Brownian motion, and σ is defined as in (2.20) #.

REMARKS

(1) The method presented above can be applied to other random hyperbolic systems.

(2) Since, for simplicity, we assume the mean value E C = η_0 is constant. This will limit the application in underwater acoustics to the propagation over the horizontal range.

(3) In theory the straitification in depth may be treated as well. The difficulty in computation lies in the deterministic, rather than stochastic part.

(4) As in theory of geometric optics, by properly choosing the asymptotic sequence, we should be able to study the statistical fluctuations near a smooth caustics, which is important in the underwater sound propagation.

(5) Since the sample solutions, in the diffusion limit, are Brownian functionals, the actual computation of statistical properties of solution may be carried out by the method of functional integrals as proposed by the author [6].

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